PHYSICAL REVIEW B **81**, 201306(R) (2010)

Correlations and beam splitters for quantum Hall anyons

Smitha Vishveshwara[1] and N. R. Cooper[2]
[1]*Department of Physics, University of Illinois at Urbana–Champaign, 1110 W. Green Street, Urbana, Illinois 61801-3080, USA*
[2]*T.C.M. Group, Cavendish Laboratory, J. J. Thomson Ave., Cambridge CB3 0HE, United Kingdom*
(Received 21 April 2010; published 27 May 2010)

We study the system of two localized anyons in the lowest Landau level and show how anyonic signatures extrapolating between antibunching tendencies of fermions and bunching tendencies of bosons become manifest in the two-particle correlations. Toward probing these correlations, we discuss the influence of a saddle potential on these anyons; we exploit analogies from quantum optics to analyze the time evolution of such a system. We show that the saddle potential can act as a beam-splitter akin to those in bosonic and fermionic systems, and can provide a means of measuring the derived anyonic signatures.

DOI: 10.1103/PhysRevB.81.201306 PACS number(s): 73.43.−f, 71.10.Pm, 05.30.Pr, 42.50.−p

The quantum statistics of bosons and fermions plays a fundamental role over a vast range of length scales and diverse physics such as the spatial distribution and energetics of electrons in atoms, certain constraints on scattering cross sections in nuclear physics, the existence of superfluids and the stability of neutron stars. Statistical signatures key to these phenomena have been explored over the past decades via analyses of two-particle correlations including in seminal studies of bosonic bunching properties by Hanbury Brown and Twiss[1] and Hong *et al.*[2] The latter, which also has its fermionic counterpart,[3–5] performs time-resolved coincidence measurements of pairs of photons incident on a beam splitter from two uncorrelated sources collected at two detectors. While these analyses have established the allowed quantum nature of particles in three dimensions, the past decade has drawn attention to the study of two-dimensional "anyons," quasiparticles which obey fractional statistics interpolating between those of fermions and bosons.[6,7] Given the current rapid experimental progress in two-dimensional systems and the keen quest for topologically ordered states which can be ascertained by the detection of anyons, a fundamental understanding of these entities analogous to that of fermions and bosons is much called for.

Here, we explore the (anti)bunching properties of a system of two noninteracting localized Abelian anyons by analyzing the behavior of specific observables and propose a means of realizing a beam splitter wherein these anyonic properties become manifest. The common wave function for these anyons by definition picks up a phase of $e^{i\pi\alpha}$ ($e^{-i\pi\alpha}$) upon a anticlockwise (clockwise) exchange of the particles.[6] The parameter α lies in the range $0<\alpha\leq 1$; $\alpha=0$ and 1 correspond to bosons and fermions, respectively. Of direct relevance to bulk quasihole excitations in the quantum Hall system[6,8–10]—a paradigm for anyonic statistics—we study a two-dimensional system of two anyons in a magnetic field projected onto the lowest Landau level (LLL).(In particular, for Laughlin states,[9] quasiholes have fractional charge $q=-e/m$ and statistics $\alpha=1/m$, where m is an odd integer.[11,12]) We find that while anyonic signatures in the LLL are subtle, they clearly extrapolate between their bosonic and fermionic counterparts. We show that the presence of a saddle potential offers a means for LLL anyons to approach one another along two incoming limbs and then propagate away along two outgoing limbs, akin to the photonic beam-splitter settings, and that analogous coincidence measurements made along the limbs can reflect our predicted anyonic signatures.

The Hamiltonian for two anyons in a perpendicular magnetic field $\boldsymbol{B}=B\hat{\mathbf{k}}$ has the decoupled form

$$\hat{H}=\frac{1}{4\mu}\left(\hat{P}_x+\frac{qB}{c}\hat{Y}_c\right)^2+\frac{1}{4\mu}\left(\hat{P}_y-\frac{qB}{c}\hat{X}_c\right)^2+\frac{1}{\mu}\left(\hat{p}_x+\frac{qB}{4c}\hat{y}_r\right)^2+\frac{1}{\mu}\left(\hat{p}_y-\frac{qB}{4c}\hat{x}_r\right)^2 \quad (1)$$

in terms of center of mass (c.o.m.) and relative variables. Here the anyons are assumed to have mass μ (which is immaterial when states are projected to the LLL) and charge q. We have chosen the symmetric gauge $\boldsymbol{A}_\gamma=(B/2)(-y_\gamma\hat{\mathbf{i}}+x_\gamma\hat{\mathbf{j}})$, $\gamma=1,2$ for each particle. The c.o.m. coordinate and momentum are given by $\boldsymbol{R}_c=(\boldsymbol{r}_1+\boldsymbol{r}_2)/2$ and $\boldsymbol{P}=\boldsymbol{p}_1+\boldsymbol{p}_2$ while the relative coordinate and momentum are given by $\boldsymbol{r}=\boldsymbol{r}_1-\boldsymbol{r}_2$ and $\boldsymbol{p}=(\boldsymbol{p}_1-\boldsymbol{p}_2)/2$. In both the c.o.m. and relative coordinate sectors, the LLL is spanned by energy degenerate angular-momentum eigenstates. The c.o.m Hilbert space is identical to that of a single-particle; angular momentum states $|n\rangle_c$ are eigenstates of angular momentum $\hbar\hat{A}^\dagger\hat{A}$ having eigenvalues $n\hbar$,where n is an integer.[13] Here, the usual commutation rules $[\hat{A},\hat{A}^\dagger]=1$ are satisfied and the components of the guiding centers have the form $\hat{X}=l(\hat{A}+\hat{A}^\dagger)/2$ and $\hat{Y}=il(\hat{A}-\hat{A}^\dagger)/2$, where $l=\sqrt{\hbar c/qB}$ is the single-particle magnetic length. In the relative coordinate sector, the anyon boundary condition is not respected by the guiding center coordinates, $\hat{x},\hat{y}$, but it is by their quadratic combinations, $\hat{a}\equiv(\hat{x}^2+\hat{y}^2)/8l^2$, $\hat{b}\equiv(\hat{x}^2-\hat{y}^2)/8l^2$, and $\hat{c}\equiv(\hat{x}\hat{y}+\hat{y}\hat{x})/8l^2$. These operators respect the $sp(1,R)$ algebra $[\hat{a},\hat{b}]=i\hat{c}$, $[\hat{b},\hat{c}]=-i\hat{a}$, and $[\hat{c},\hat{a}]=i\hat{b}$.[14] The relative coordinate Hilbert space consists of irreducible representations of this algebra $|k,\alpha\rangle_r$, where k is an integer, and correspond to eigenstates of the angular momentum $\hat{L}=\hbar(2\hat{a}-1/2)$ having eigenvalues $(2k+\alpha)\hbar$. These angular momentum eigenstates satisfy $a|k,\nu\rangle=(k+\mu)|k,\nu\rangle$ and $\gamma|k,\nu\rangle=\mu(\mu-1)|k,\nu\rangle$ where $\mu\equiv\nu/2+1/4$ and $\gamma\equiv a^2-b^2-c^2$ is Casimir operator.[10,14]

Localized anyons can be composed as linear combinations of the degenerate LLL states. These states can be decomposed into product states of localized states centered at the dimensionless c.o.m. coordinates $Z=(z_1+z_2)/2$ and relative coordinates $z=z_1-z_2$, where the individual anyons are centered at $z_\gamma=(x_\gamma+iy_\gamma)/l$, $\gamma=1,2$. These localized states, which are not equivalent to coherent states except for bosons and fermions, have been shown to have the form[10]

$$|Z\rangle_c = e^{-|Z|^2/2}\sum_{n=0}^{\infty}\frac{(Z^*)^n}{\sqrt{n!}}|n\rangle_c, \tag{2}$$

$$|z\rangle_\alpha = N_{\alpha,z}\sum_{k=0}^{\infty}\frac{(z^*/2)^{2k+\alpha}}{\sqrt{\Gamma(2k+\alpha+1)}}|k,\alpha\rangle_r. \tag{3}$$

Here, we express the normalization $N_{\alpha,z}$, which in itself contains information on statistics, in terms of a sum of two confluent hypergeometric functions as $2|N_{\alpha,z}|^{-2}\Gamma(1+\alpha) = (|z|/2)^{2\alpha}[M(1,1+\alpha,|z|^2/4)+M(1,1+\alpha,-|z|^2/4)]$. The convention chosen for the relative coordinate localized state explicitly respects the anyonic boundary condition in picking up the desired phase under the exchange action $z\to ze^{i\pi}$. Our restriction on the α range $0<\alpha\leq 1$ ensures that the localized anyon wave function is regular at the origin and that the focus is purely on statistical interactions.

The localized states in Eq. (3) are consistent with those for fermions and bosons. There (anti)symmetrization is achieved by constructing

$$|z\rangle_{1/0} = e^{|z|^2/8}N_{1/0,z}[|z\rangle_d \mp |-z\rangle_d], \tag{4}$$

where $|z\rangle_d$ refers to the localized state for distinguishable particles, analogous to Eq. (2) but with $Z\to z/2$. For localized states, the probability density must be symmetrically peaked close to both relative coordinates z and $-z$ for all indistinguishable particles; this can be ascertained by analyzing the anyon state of Eq. (3). The construction of Eq. (4) explicitly shows that the fermion/boson boundary conditions allow only odd/even angular momentum states in the relative coordinate localized state decomposition. By evaluating the overlap between $|z\rangle_d$ and $|-z\rangle_d$, one finds that the normalization constant has limiting forms $|N_{1,z}|^{-2}=\sinh(|z|^2/4)$ and $|N_{0,z}|^{-2}=\cosh(|z|^2/4)$, respectively.

One of the most direct measures of quantum statistics and related (anti)bunching behavior is the average guiding center separation squared, $\langle\hat{\boldsymbol{r}}^2\rangle\equiv\langle\hat{x}^2+\hat{y}^2\rangle$. It is well known that for any generic system of spinless fermions/bosons, (anti)symmetrization leads to this average separation being greater/smaller than the value for distinguishable particles.[15,16] Generally, this statistical effect becomes most pronounced at smaller separation while at larger separation, statistical correlations decay in a manner characteristic to the particular system. Here, to quantify this statistical effect, we define a bunching parameter

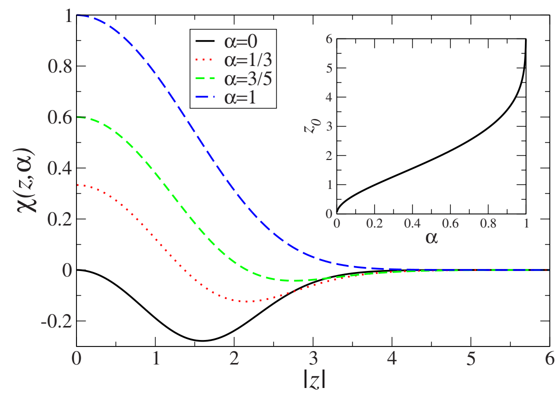

FIG. 1. (Color online)The bunching parameter as a function of the dimensionless distance between particles $|z|$ for different values of anyonic phase α. Curves from the topmost along the y axis correspond to values $\alpha=1$, $\alpha=3/5$, $\alpha=1/3$, and $\alpha=0$. Inset: the value z_0 as a function of α, where $\chi(|z_0|,\alpha)=0$ determines the crossover between antibunching and bunching behavior.

$$\chi(|z|,\alpha) \equiv \frac{1}{4l^2}[{}_\alpha\langle z|\hat{\boldsymbol{r}}^2|z\rangle_\alpha - {}_d\langle z|\hat{\boldsymbol{r}}^2|z\rangle_d], \tag{5}$$

where the factor of $4l^2$ is a matter of convention. A (positive) negative value of χ implies (anti)bunching in comparison with distinguishable particles.

We now evaluate the bunching parameter for localized LLL anyons. For distinguishable particles, we have the expected form ${}_d\langle z|\hat{\boldsymbol{r}}^2|z\rangle_d=(|z|^2+2)l^2$, where the nonzero minimum value reflects the finite width associated with the minimum uncertainty in guiding center positions x and y characteristic of states in the LLL. For fermions and bosons, $\langle\hat{\boldsymbol{r}}^2\rangle$ can be directly evaluated using the definition in Eq. (4). The bunching parameter takes the forms $\chi(z,1)=(|z|^2/4)[\coth(|z|^2/4)-1]$ for fermions and $\chi(z,0)=(|z|^2/4)[\tanh(|z|^2/4)-1]$ for bosons. In keeping with expectations, the bunching parameter is always positive/negative for fermions/bosons and decays exponentially toward zero for large $|z|$.

For anyons, the desired expectation values can be evaluated by using $\hat{\boldsymbol{r}}^2=8l^2\hat{a}$, and the eigenstate property $\hat{a}|k,\alpha\rangle_r=(k+\alpha/2+1/4)|k,\alpha\rangle_r$, and hence

$$ {}_\alpha\langle z|8\hat{a}-2|z\rangle_\alpha = |N_{\alpha,z}|^2\sum_{k=0}^{\infty}\frac{(8k+4\alpha)\left(\frac{|z|^2}{4}\right)^{2k+\alpha}}{\Gamma(2k+\alpha+1)} = \frac{4\alpha\left[M\left(1,\alpha,\frac{|z|^2}{4}\right)+M\left(1,\alpha,-\frac{|z|^2}{4}\right)\right]}{M\left(1,1+\alpha,\frac{|z|^2}{4}\right)+M\left(1,1+\alpha,-\frac{|z|^2}{4}\right)}. \tag{6}$$

Figure 1 shows the trend exhibited by the bunching parameter obtained from Eq. (6). Quite remarkably, the value of χ at $|z|^2=0$ (which is not physically accessible in quantum Hall samples) directly reflects the statistical parameter; $\chi(0,\alpha)=\alpha$. For the limiting case of the fermion, as a function of $|z|$, $\chi(|z|,1)$ begins at a value of unity and then decays to zero in a monotonic fashion. For the bosonic case, $\chi(|z|,0)$ always remains negative, beginning at zero decreasing to a minimum value and then rising to taper toward zero. The intermediate anyonic values of α interpolate between these two limiting behaviors. For all anyons, $\chi(|z|,\alpha)$ begins at α, decreases below zero, reaches a minimum and finally tapers toward the zero. Hence, the bunching parameter shows that all anyons exhibit antibunching at short length scales and bunching at long scales and that the trend evolves continuously as a function of α. This result is surprising in that one might expect $\alpha<1/2$ to be bosonlike and $\alpha>1/2$ to be fermionlike. The behavior of $\chi(|z|,\alpha)$ shown in Fig. 1 and its connection to fractional statistics forms the heart of our results.

While the correlations discussed above bear distinct signatures of statistics, they are static in nature due to the projection to the LLL. In practice, probing these correlations requires endowing them with dynamics via the application of appropriate potentials that lift the LLL degeneracy. Here we propose the application of a saddle potential whose effect on each particle can be described by $\hat{H}_s=\Sigma_{\gamma=1,2}U\hat{x}_\gamma\hat{y}_\gamma$, $U>0$, where Ul^2 is much smaller than the Landau-level spacing thus retaining the LLL projection. In terms of LLL eigenstate solutions for a single particle,[17] it has been shown that the saddle potential acts as a beam splitter in that particles approaching the origin along the x axis tend to scatter either along the positive or the negative y axis. Moreover, for two particles, the potential has the advantage of being separable in terms of the relative and center-of-mass motion,[18] hence preserving the decoupling of these degrees of freedom, and of respecting the anyon boundary conditions. The saddle potential, when projected to the LLL,[10,13] can be expressed as

$$\hat{H}_s^P=\frac{1}{2}iUl^2[\hat{A}^2-(\hat{A}^\dagger)^2]+2Ul^2\hat{c}. \tag{7}$$

As for the single-particle case, eigenstates of this Hamiltonian correspond to scattering states in the relative and center-of-mass sectors, and that anyonic statistics translates to scattering phase shifts in the relative sector.[18] Here we show that for pairs of localized particles traveling along opposite limbs of the saddle on the x axis, the choice of propagation along the y axis directly reflects the correlations similar to those shown in Fig. 1.

To gain an insight on the propagation of localized states along saddle potentials, we present an analysis of single-particle physics exploiting analogies in quantum optics[19] which can also be applied for the c.o.m. behavior. The associated localized c.o.m. state in Eq. (3) has the coherent-state form $|Z\rangle_c=\exp(Z\hat{A}^\dagger-Z^*\hat{A})|0\rangle_c\equiv\hat{D}(Z)|0\rangle_c$. In the Schrödinger picture, we can consider the time evolution of the coherent state due to the saddle potential: $|Z(t)\rangle_c=e^{-i\hat{H}_s^P t/\hbar}|Z\rangle_c$. In the language of quantum optics, the time-evolution operator has the form of the squeeze operator $\hat{S}(\xi)=\exp[\xi(\hat{A}^\dagger)^2/2-\xi^*\hat{A}^2/2]$, where for the c.o.m. sector we have $\xi=-Utl^2/\hbar$. The squeeze parameter $\xi\equiv re^{i\phi}$ corresponds to squeezing along the direction ϕ with associated aspect ratio r; here the squeeze is of magnitude $Utl^2/\hbar$ along the x axis. We can now invoke the identity $\hat{S}(\xi)\hat{D}(Z)=\hat{D}(Z\cosh r+Z^*e^{i\phi}\sinh r)\hat{S}(\xi)$ and the fact that $\hat{D}(\beta)\hat{S}(\xi)|0\rangle$ represents a squeezed state having squeeze parameter ξ centered at β.[19] Hence, the time-evolved coherent state flattens along the y axis and its center follows the trajectory $(Xe^{-Utl^2/\hbar},Ye^{Utl^2/\hbar})$, where (X,Y) is the initial position of the coherent state. Consistent with semiclassical dynamics along equipotentials of a saddle potential, the center obeys $X(t)Y(t)$ being a constant. Furthermore, any state having the initial condition $Y(t=0)>0$ evolves asymptotically toward $Y(t\to\infty)\to+\infty$ and likewise for the lower quadrant.

For the relative motion of anyons the analysis presented above cannot be directly applied as the associated $sp(1,R)$ algebra is rather involved. However, we surmise a few common features and explicitly derive time-evolved expectation values of relevant observables. Given that the initial-state probability density for the relative coordinate is peaked at z and $-z$, over time, we expect it to asymptotically be distributed in the upper and lower quadrants in a manner which depends on the statistics of the particles. The functioning of the saddle potential as a beam splitter is best seen when two localized state anyons are placed along or close to the x axis, diametrically across one another with respect to the saddle-point origin. As a function of time, the particles approach one another and then get deflected along the y axis. Whether or not they travel in the same direction (along either the positive or negative y direction) or in opposite directions depends on the magnitude of $\langle\hat{Y}^2\rangle$ compared to that of $\langle\hat{y}^2\rangle$. In fact, the quantity analogous to those measured in photonic and electronic beam splitters is $\langle\hat{y}_1\hat{y}_2\rangle=\langle\hat{Y}^2-\hat{y}^2/4\rangle$; a positive/negative value of $\langle\hat{y}_1\hat{y}_2\rangle$ indicates that the anyons traveled out along the same/opposite limbs, thus exhibiting bunching/antibunching behavior. These correlations are analogous to those between reflected and transmitted currents in electronic beam splitters.[4,5]

The desired time-evolved expectation values are most easily evaluated in the Heisenberg representation. From the commutation relations of $\hat{a}$, $\hat{b}$, and $\hat{c}$,[14] one finds the Heisenberg equations of motion $d\hat{a}/dt=-2Ul^2\hat{b}$, $d\hat{b}/dt=-2Ul^2\hat{a}$. The solutions of these equations yield $\hat{x}^2(t)=e^{-2Utl^2/\hbar}\hat{x}^2(0)$ and $\hat{y}^2(t)=e^{2Utl^2/\hbar}\hat{y}^2(0)$ for the relative coordinates. Similarly, and consistent with the above discussion, one finds $\hat{X}^2(t)=e^{-2Utl^2/\hbar}\hat{X}^2(0)$ and $\hat{Y}^2(t)=e^{2Utl^2/\hbar}\hat{Y}^2(0)$ for the c.o.m. coordinates. Evaluating the expectation values of these operators for the initial state described by Z and z, we find that the correlator for the relative position along the y axis is

$$\langle\hat{y}_1\hat{y}_2\rangle=l^2e^{2Utl^2/\hbar}\left[\mathrm{Im}[Z]^2-\frac{1}{4}\mathrm{Im}[z]^2-\frac{1}{2}\chi+\delta\right], \tag{8}$$

where $\chi(|z|,\alpha)$ is the bunching parameter introduced in Eq. (5). The function $\delta(z,\alpha)$ is a small correction, which has the maximum magnitude of 0.018, vanishes for $z=0$ and arises due to the deviation of localized states from coherent states. Hence, for anyons placed on the x axis, the sign of $\langle\hat{y}_1\hat{y}_2\rangle$, or equivalently, whether the particles went into the same or opposite limbs, is determined by the statistics and the bunching parameter for the initial conditions. Given the exponential dependence of $\langle \hat{y}_1 \hat{y}_2 \rangle$ on time, the saddle potential acts as a beam splitter whose read out amplifies initial correlations. As discussed above and shown in Fig. 1, for bosons/fermions, χ is always negative/positive, recovering the well-known result that statistically correlated bosons/fermions travel along the same/different limbs. For anyons, the sign of the bunching parameter directly determines whether pairs of particles go into the same limb or opposite limbs. Thus, for particles initially placed close together, particles propagate into opposite limbs as do fermions while those placed further out propagate into the same limbs; the transition point between these two dramatically different possibilities depends on the fractional statistics parameter. A clear-cut signature of anyons is that, unlike for fermions and bosons, both possibilities are present and accessible by tuning initial conditions.

Experimentally, for a direct implementation one can envisage initializing two quasiparticles in the quantum Hall bulk in two locally created potential minima, as has been observed for single quasiparticles,[20] applying a saddle potential and collecting quasiparticles along receiver limbs by way of other local potential traps. Such local state preparation would allow a direct mapping of the bunching parameter χ. However, this requires some further advances in the experimental control of the quantum Hall bulk.

Alternatively, partial information on χ can be obtained with existing experimental technology by exploiting the beautiful geometries and methods that use quantum Hall edge states as sources of quasiparticles and that have already shown beam-splitter physics and the fermionic statistics of electrons.[4,21] A specific experiment faithful to previously discussed settings[4,22] involves two edge states meeting at a quantum-point contact (QPC), which is locally a saddle potential. Quasiparticles impinging on the QPC scatter into the outgoing channels in a manner that depends on their mutual statistics. Here we take a simplified model of heavily diluted beams, such that any interactions in the QPC are pairwise. For a QPC tuned to 50% transmission, the quasiparticles are incident with $\mathrm{Im}(Z)=\mathrm{Im}\, z=0$ in Eq. (8), so exit in the same ($\langle \hat{y}_1 \hat{y}_2 \rangle > 0$, bunching) or different ($\langle \hat{y}_1 \hat{y}_2 \rangle < 0$, antibunching) channels depends on the sign of $\chi(x)-2\delta$. Here x is the initial separation of the incoming particles determined by the spatial extent of the QPC and the average separation between quasiparticles in each beam. The separation x could thus be changed by small variations in magnetic field as well as degree of dilution of the beams. A measure of χ as a function of x could then be obtained from current-correlation measurements between the outgoing particles. While a numerical analysis of this effect is clearly desirable (to allow a measurement of α), this requires the development of a detailed model for the diluted beams and the QPC which is beyond the scope of the present work.

In conclusion, we have presented fundamental correlations characterizing LLL anyons and distinguishing them from their fermionic and bosonic counterparts. We have proposed the application of a saddle potential as a means of realizing a quantum Hall beam splitter that can display these correlations and associated direct signatures of fractional statistics.

We would like to acknowledge M. Heiblum, P. Kwiat, and D. Sen for illuminating discussions. This work was supported by the NSF under Grant No. DMR 06-44022 CAR, by the Center for Advanced Studies at UIUC, and by EPSRC under Grant No. EP/F032773/1.

[1] R. Hanbury Brown and R. Q. Twiss, Philos. Mag. **45**, 663 (1954); Nature (London) **177**, 27 (1956).
[2] C. K. Hong, Z. Y. Ou, and L. Mandel, Phys. Rev. Lett. **59**, 2044 (1987).
[3] R. Loudon, Phys. Rev. A **58**, 4904 (1998).
[4] M. Henny *et al.*, Science **284**, 296 (1999); I. Neder *et al.*, Nature (London) **448**, 333 (2007).
[5] W. D. Oliver *et al.*, Science **284**, 299 (1999); M. Kindermann, Nature (London) **448**, 262 (2007), and reference therein.
[6] J. Leinaas and J. Myrheim, Nuovo Cimento Soc. Ital. Fis., B **37**, 1 (1977).
[7] F. Wilczek, Phys. Rev. Lett. **48**, 1144 (1982).
[8] B. I. Halperin, Phys. Rev. Lett. **52**, 1583 (1984).
[9] R. B. Laughlin, in *The Quantum Hall Effect*, edited by R. E. Prange and S. M. Girvin (Springer-Verlag, New York, 1987).
[10] H. Kjønsberg and J. Leinaas, Int. J. Mod. Phys. A **12**, 1975 (1997).
[11] R. B. Laughlin, Phys. Rev. Lett. **50**, 1395 (1983); Rev. Mod. Phys. **71**, 863 (1999).
[12] D. Arovas, J. R. Schrieffer, and F. Wilczek, Phys. Rev. Lett. **53**, 722 (1984).
[13] J. K. Jain, *Composite Fermions* (Cambridge University Press, Cambridge, 2007), and references therein.
[14] T. H. Hansson, J. M. Leinaas, and J. Myrheim, Nucl. Phys. B **384**, 559 (1992).
[15] See, e.g., G. Baym, Acta Phys. Pol. B **29**, 1839 (1998).
[16] D. J. Griffiths, *Introduction to Quantum Mechanics* (Pearson Education, Upper Saddle River, NJ, 2005).
[17] H. A. Fertig and B. I. Halperin, Phys. Rev. B **36**, 7969 (1987).
[18] A. Matthews and N. Cooper, Phys. Rev. B **80**, 165309 (2009).
[19] See, e.g., H. A. Bachor and T. C. Ralph, *A Guide to Experiments in Quantum Optics* (Wiley-VCH, Weinheim, Germany, 2004); P. D. Drummond and Z. Ficek, *Quantum Squeezing* (Springer-Verlag, Berlin, 2004).
[20] J. Martin *et al.*, Science **305**, 980 (2004).
[21] Y. Ji *et al.*, Nature (London) **422**, 415 (2003).
[22] P. Bonderson, K. Shtengel, and J. K. Slingerland, Ann. Phys. **323**, 2709 (2008).